Normality and Automata

Olivier Carton

LIAFA
Université Paris Diderot & CNRS

Join work with Verónica Becher and Pablo Heiber
(Universidad de Buenos Aires & CONICET)

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Outline

Normality

Compressibility

- One-way transducers
- Two-way transducers

Selection

- Prefix selection
- Suffix selection

Expansion of real numbers

Fix an integer base $b \geqslant 2$. The alphabet is $A = \{0, 1, \dots, b-1\}$.

- if $b = 2$, $A = \{0, 1\}$,
- if $b = 10$, $A = \{0, 1, 2, \dots, 9\}$.

Each real number $\xi \in [0, 1)$ has an expansion in base b:
$x = a_1 a_2 a_3 \cdots$ where $a_i \in A$ and

$$\xi = \sum_{k \geq 1} \frac{a_k}{b^k}.$$

In the rest of this talk:

$$\begin{array}{rcl} \text{real number } \xi \in [0,1) & \longleftrightarrow & \text{infinite word } x \in A^\omega \\ 1/3 & \longleftrightarrow & 010101\cdots = (01)^\omega \\ \pi/4 & \longleftrightarrow & 1100100100001111\cdots \end{array}$$

Normality (Borel 1909)

The number of occurrences of a word u in a word w is

$$\mathrm{occ}(w,u) = |\{i : w[i..i+|u|-1] = u\}|$$

An infinite word $x \in A^\omega$ (resp. a real number ξ) is simply normal (in base b) if for any $a \in A$,

$$\lim_{n\to\infty} \frac{\mathrm{occ}(x[1..n],a)}{n} = \frac{1}{b}.$$

An infinite word $x \in A^\omega$ (resp. a real number ξ) is normal (in base b) if for any $u \in A^*$,

$$\lim_{n\to\infty} \frac{\mathrm{occ}(x[1..n],u)}{n} = \frac{1}{b^{|u|}}.$$

In base $b = 2$, this means

- the frequencies in x of the 2 digits 0 and 1 are $1/2$,
- the frequencies in x of the 4 words $00, 01, 10, 11$ are $1/4$,
- the frequencies in x of the 8 words $000, 001, \dots, 111$ are $1/8$,
- ...

Examples

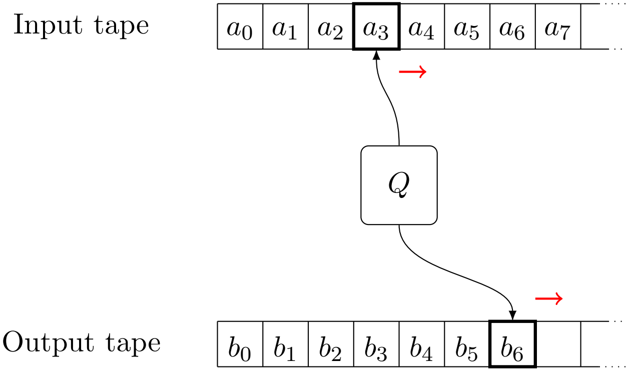
Theorem (Borel 1909)

Almost all real numbers are normal, that is, the measure of the set of normal numbers in $[0, 1)$ *is* 1.

Examples

- the infinite word $(001)^\omega = 0010010\cdots$ is not simply normal in base 2,
- the infinite word $(01)^\omega = 01010\cdots$ is simply normal in base 2 but it is not normal,
- the Champernowne word $012345678910111213\cdots$ is normal in base 10.
- the Champernowne word $011011100101110111\cdots$ is normal in base 2.

Transducers

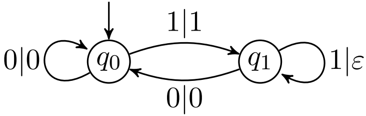


Transitions $p \xrightarrow{a|v} q$ for $a \in A$, $v \in B^*$.

Examples

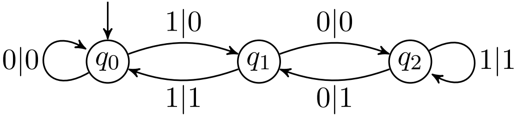
A transducer is an automaton $\mathcal{T} = \langle Q, A, B, \Delta, I, F \rangle$ where Δ is a finite set of transitions $p \xrightarrow{a|v} q$ where $a \in A$ and $v \in A^*$.

Example (Compression of blocks of consecutive 1)



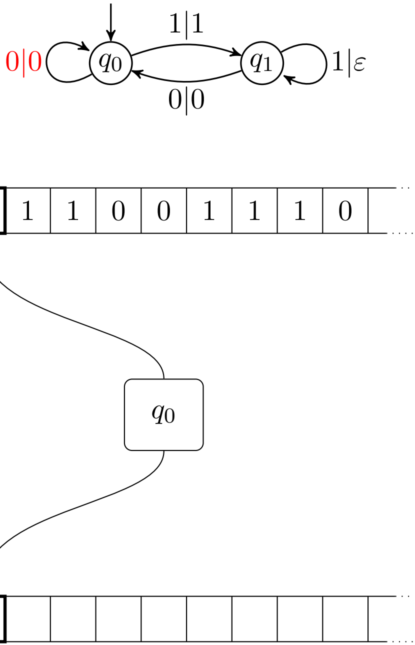
If the input is $010011000111\cdots$, the output is $01001000100\cdots$.

Example (Division by 3 in base 2)



If the input is $(01)^\omega$, the output is $(000111)^\omega$.

Example



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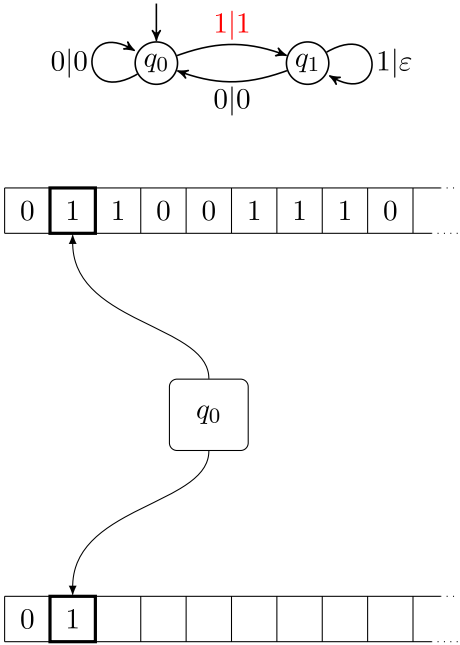

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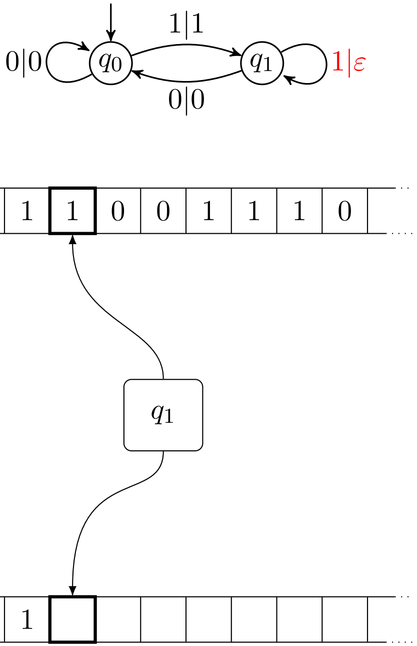

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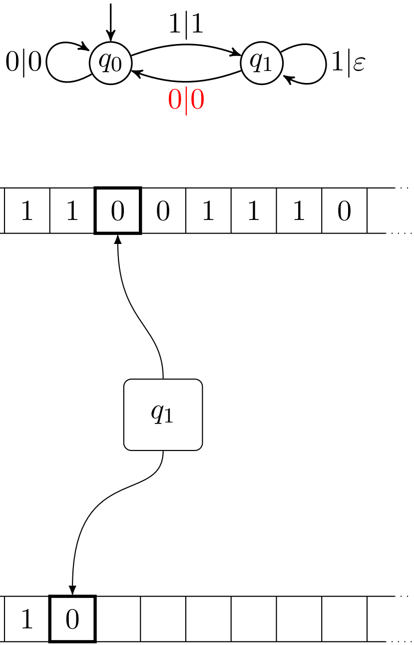

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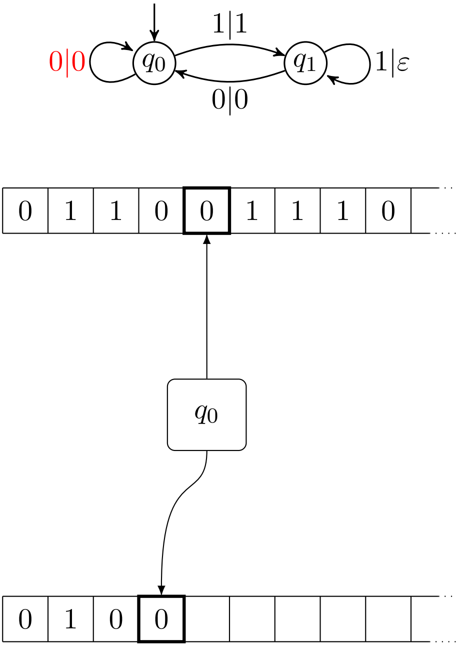

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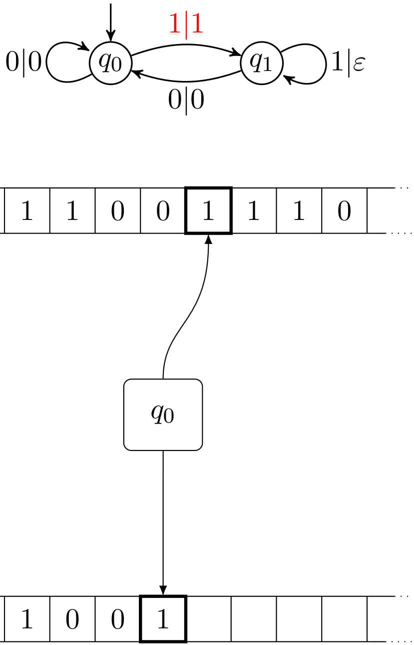

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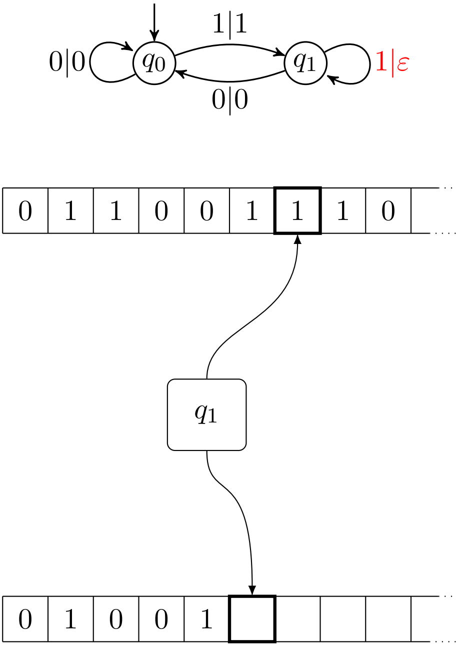

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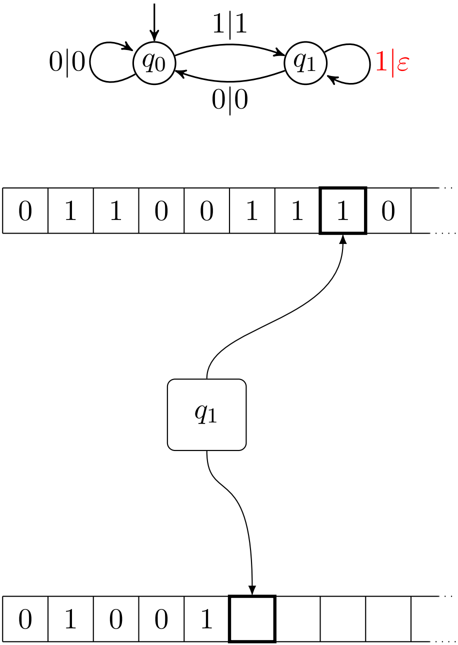

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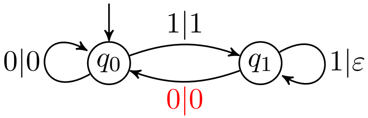
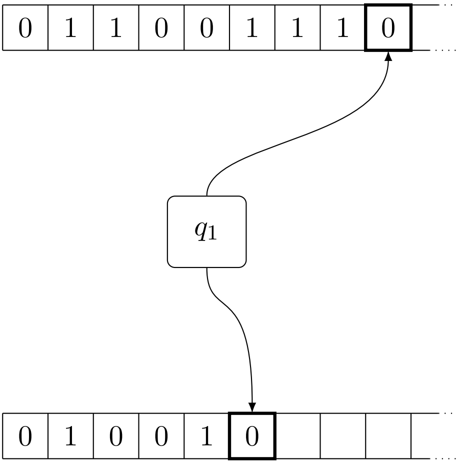




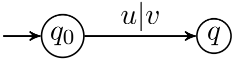
Transducers as compressors

An infinite word $x = a_1 a_2 a_3 \cdots$ is *compressible* by a transducer if there is an accepting run $q_0 \xrightarrow{a_1|v_1} q_1 \xrightarrow{a_2|v_2} q_2 \xrightarrow{a_3|v_3} q_3 \cdots$ satisfying

$$\liminf_{n\to\infty} \frac{|v_1 v_2 \cdots v_n| \log |B|}{|a_1 a_2 \cdots a_n| \log |A|} < 1.$$

Different notions of compressors

- the function $x \mapsto T(x)$ is one-to-one
- deterministic lossless: the map $u \mapsto (v, q)$ is one-to-one



- the function $x \mapsto T(x)$ is bounded-to-one
  There is a constant K such that $|\{x : T(x) = y\}| \leqslant K$.

Characterization of normal words

Theorem (Many people)

An infinite word is normal if and only if it cannot be compressed by deterministic lossless transducers.

- Schnorr and Stimm (1971)
  non-normality $\Leftrightarrow$ finite-state martingale success
- Dai, Lathrop, Lutz and Mayordomo (2004)
  compressibility $\Leftrightarrow$ finite-state martingale success
  normality $\Rightarrow$ no martingale success
- Bourke, Hitchcock and Vinodchandran (2005)
  non-normality $\Rightarrow$ martingale success
- Becher and Heiber (2013)
  non-normality $\Leftrightarrow$ compressibility (direct)
- Becher, Carton and Heiber
  generalized to bounded-to-one

Randomness

Randomness can be characterized as non-compressibility:

$$\liminf_{n\to\infty} \mathcal{H}(x[1..n]) - n > -\infty$$

where $\mathcal{H}$ is the prefix Kolmogorov complexity of the finite word w.

Normal infinite words are the random words for automata.

Turing may compress some normal words (Champernowne's).
What is the real power needed to compress a normal word ?

Ingredients

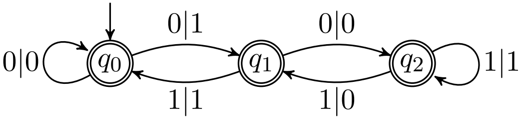
Shannon (1958)

- frequency of u different from $b^{-|u|}$ implies non maximum entropy
- non-maximum entropy implies compressibility

Huffman (1952)

- simple greedy implementation of Shannon's general idea
- implementation by a finite state tranducer

Deterministic vs Non-Deterministic transducers



Multiplication by 3 in base 2

Theorem

Non-deterministic bounded-to-one transducers cannot compress normal infinite words.

Counter transducers

- the transducer uses k-counters with integer values that can be incremented, decremented and tested for zero
- real-time restriction: incrementation and decrementation can only occur when a input symbol is processed

Theorem

Bounded-to-one counter transducers cannot compress normal infinite words.

Non-real-time two-counter machines are Turing complete.

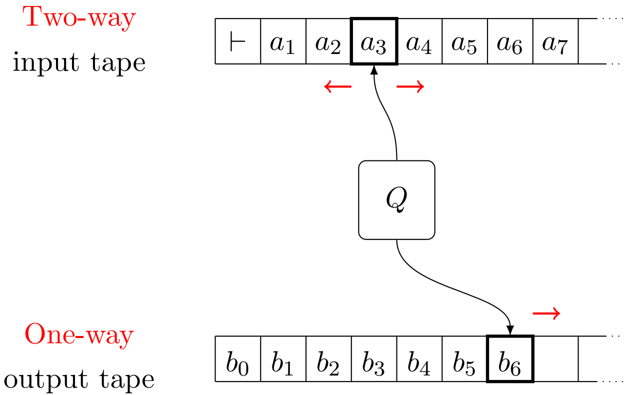
Summary of the results

	det	non-det	non-rt
finite-state	N	N	N
1 counter	N	N	N
≥ 2 counters	N	N	T
1 stack	?	C	C
1 stack + 1 counter	C	C	T

where

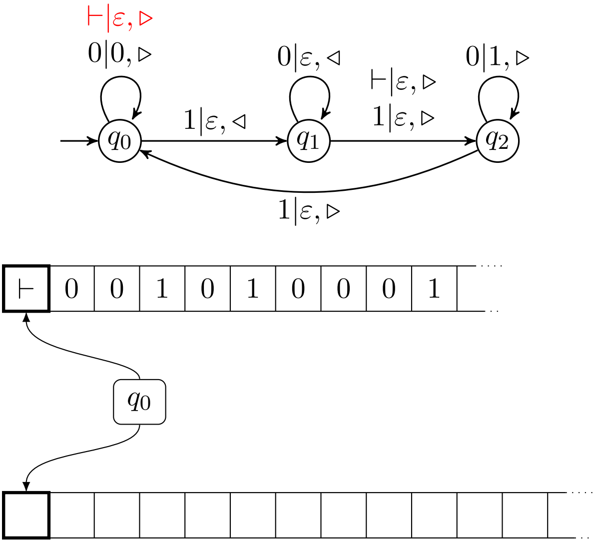
- N means *cannot compress normal words*
- C means *can compress some normal word*
- T means *is Turing complete* and thus can compress.

Two-way transducers



Transitions $p \xrightarrow{a|v,d} q$ for $a \in A$, $v \in B^*$ and $d \in \{\triangleleft, \triangleright\}$.

Example: $0^{n_0}10^{n_1}10^{n_2}1\cdots \mapsto 0^{n_0}1^{n_0}0^{n_1}1^{n_1}0^{n_2}1^{n_2}\cdots$



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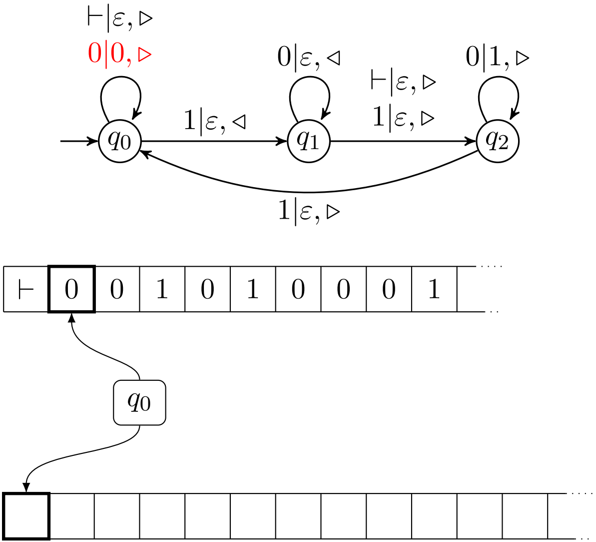

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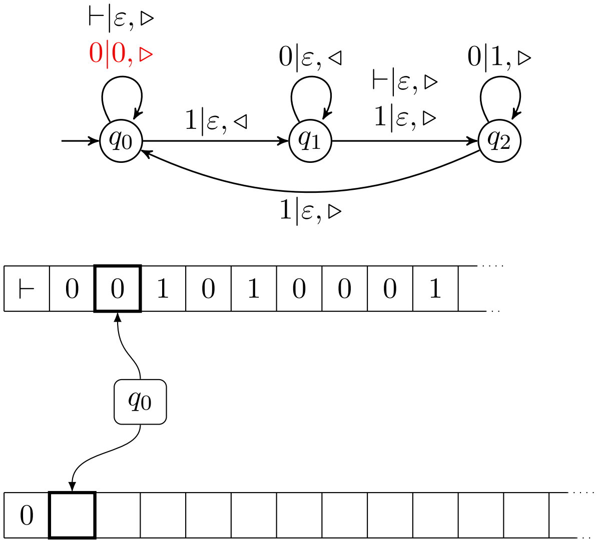

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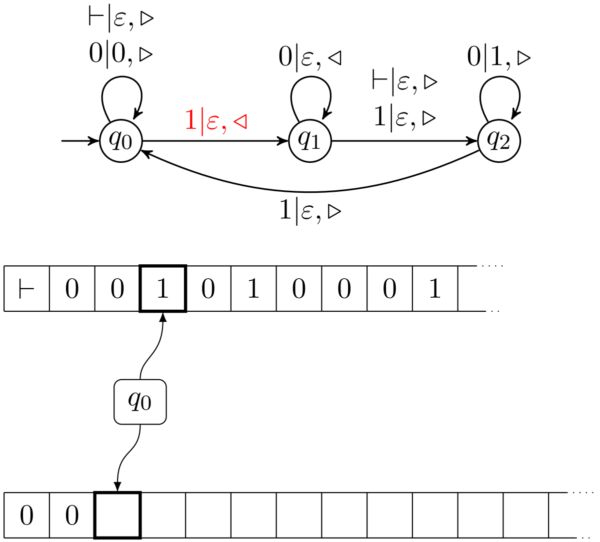

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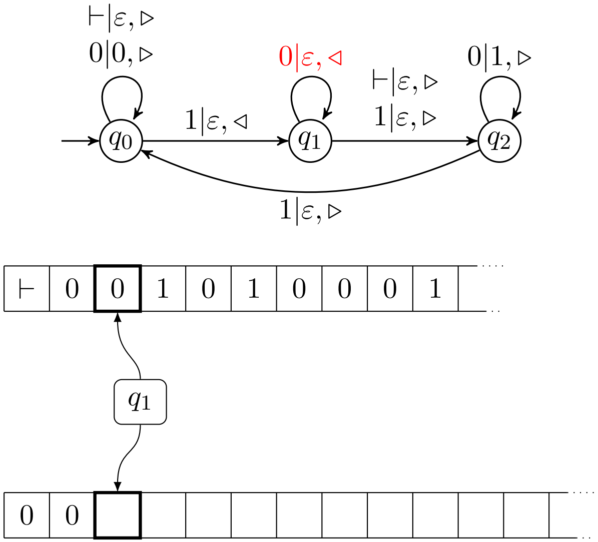

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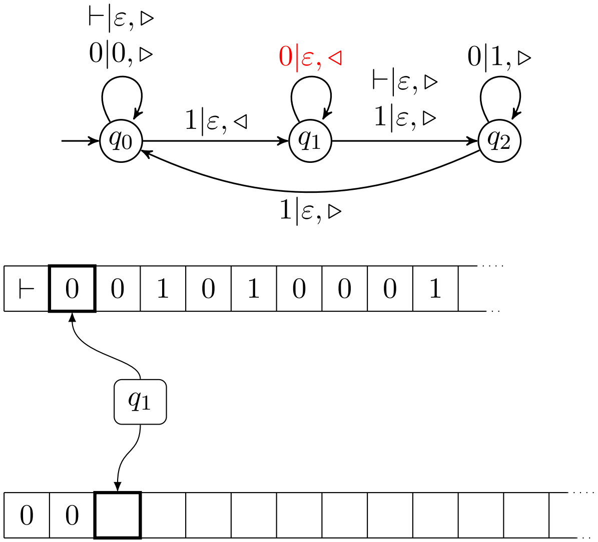

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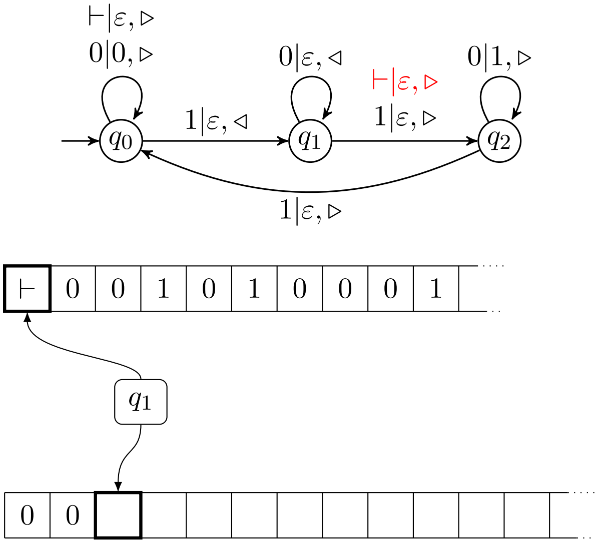

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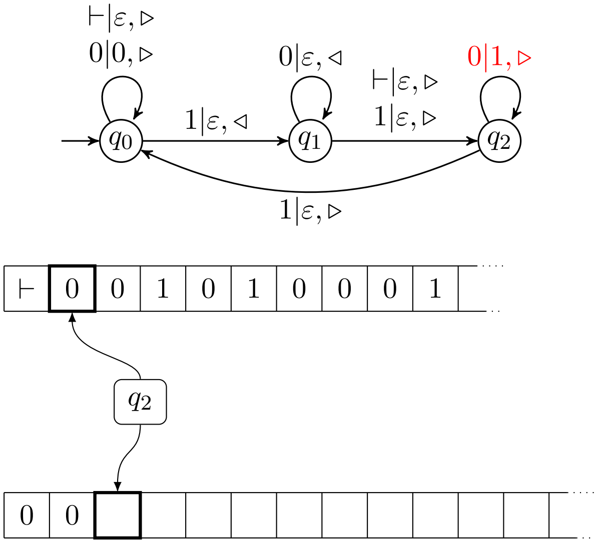

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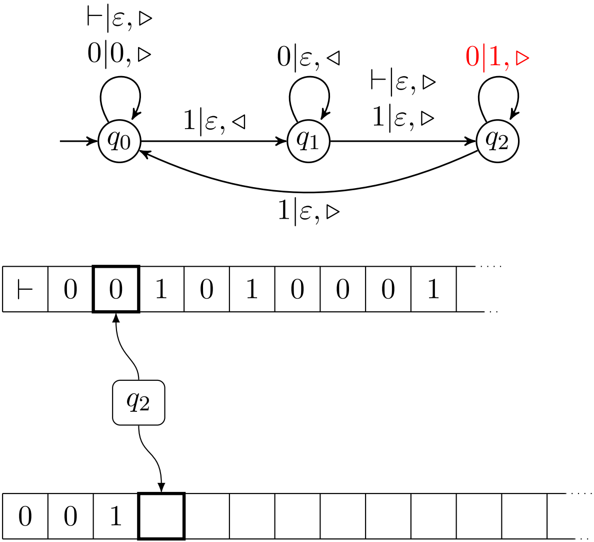

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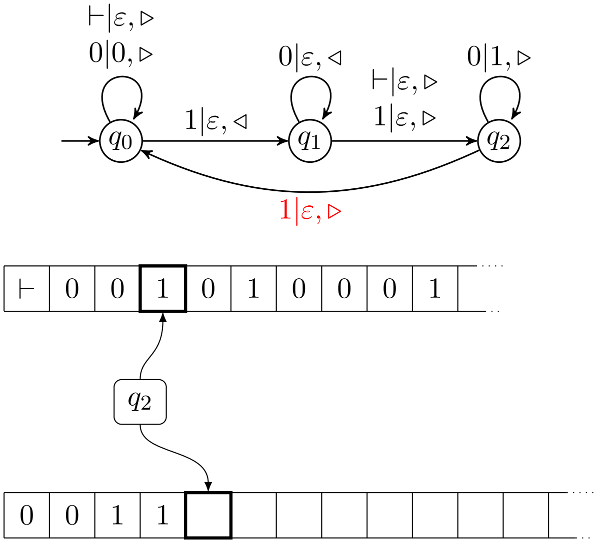

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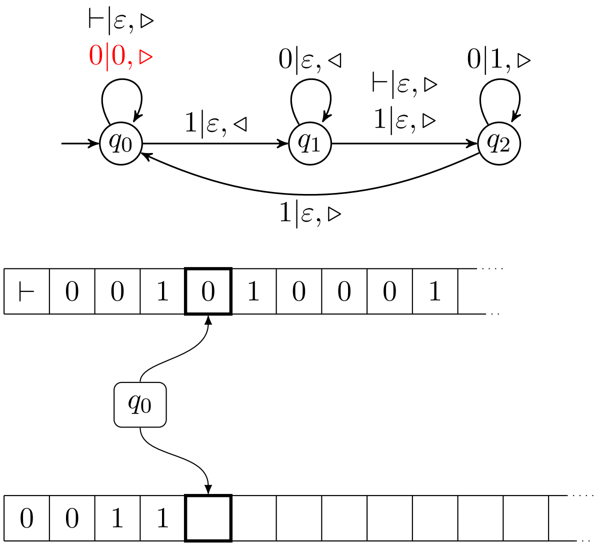

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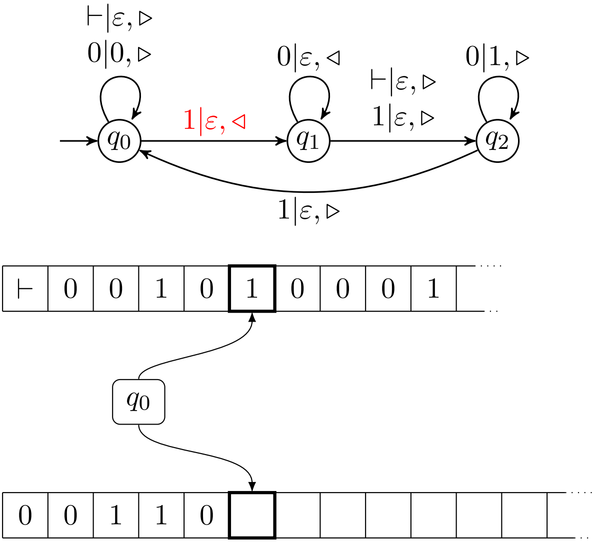

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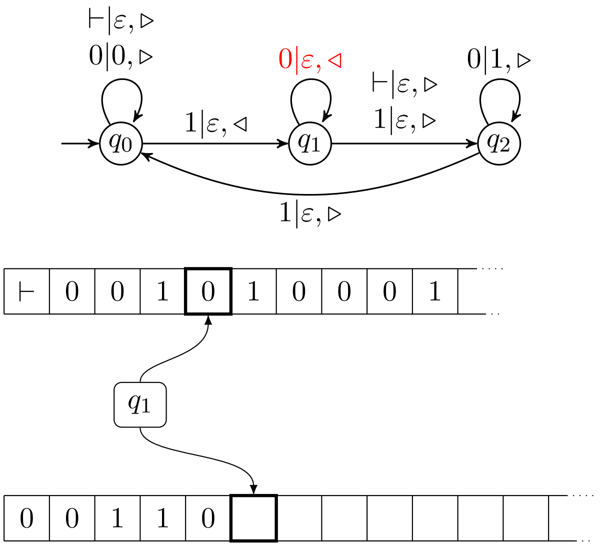

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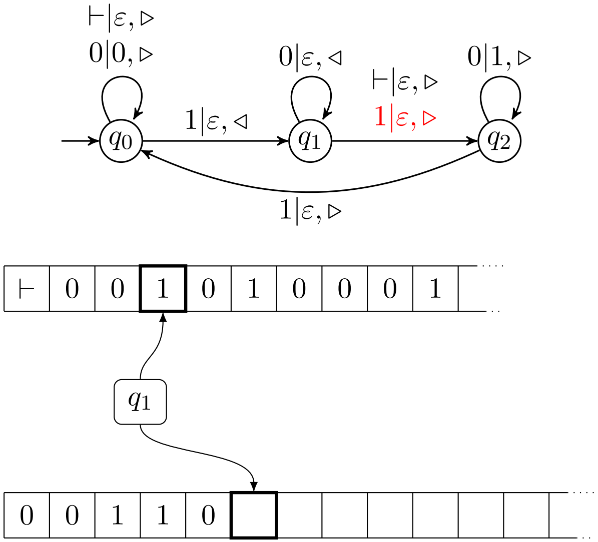

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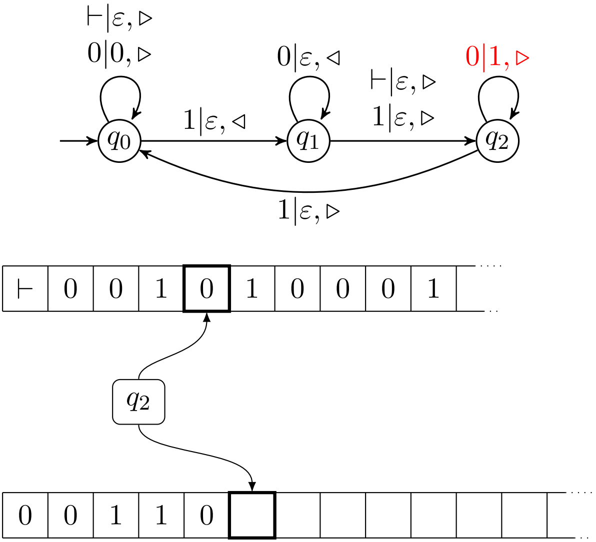

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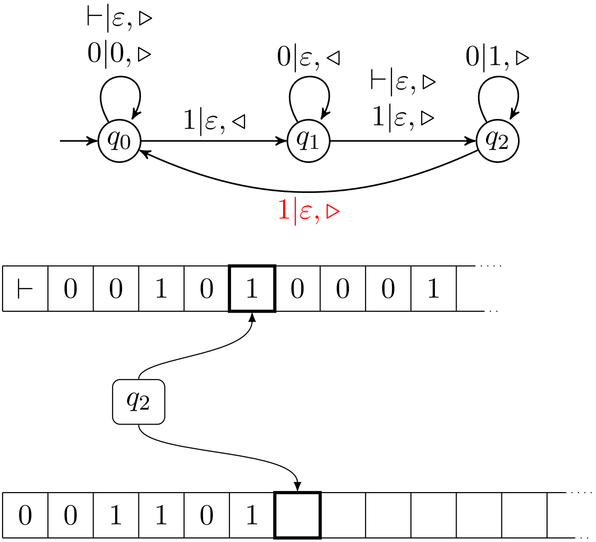

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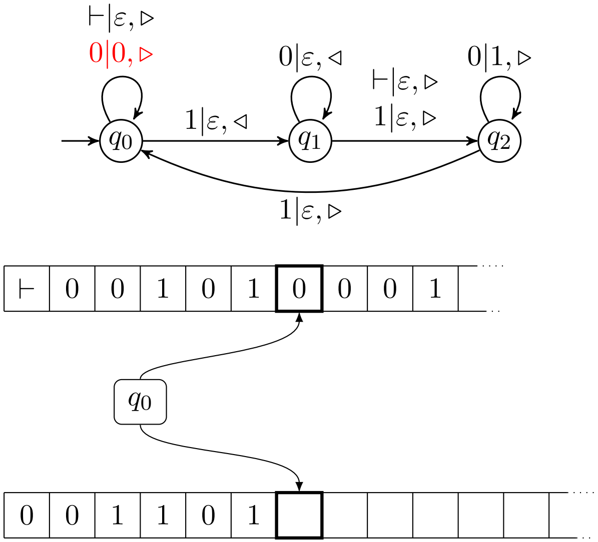

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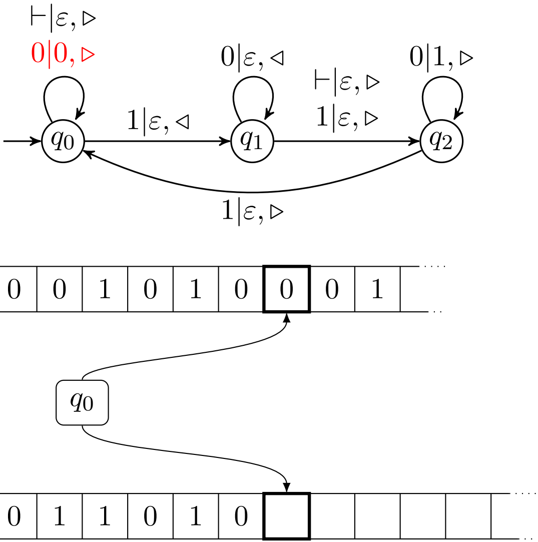

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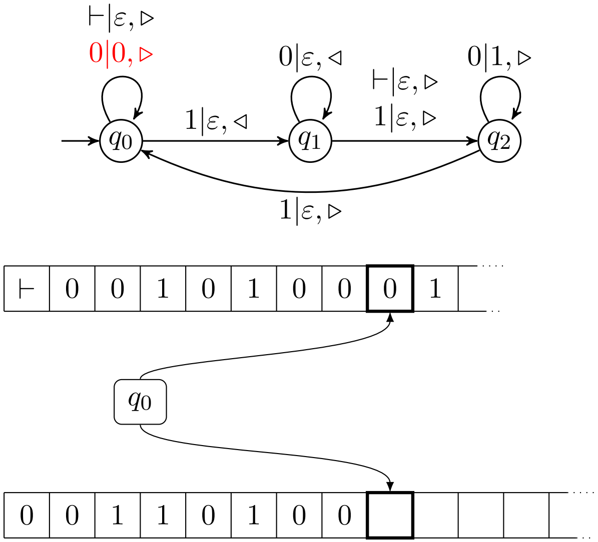

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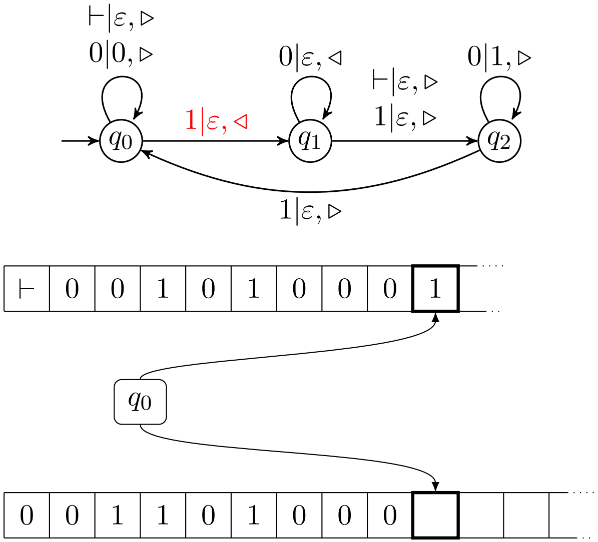

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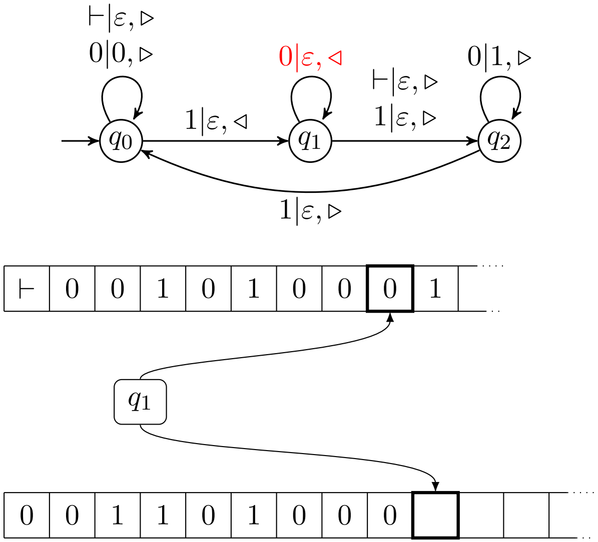

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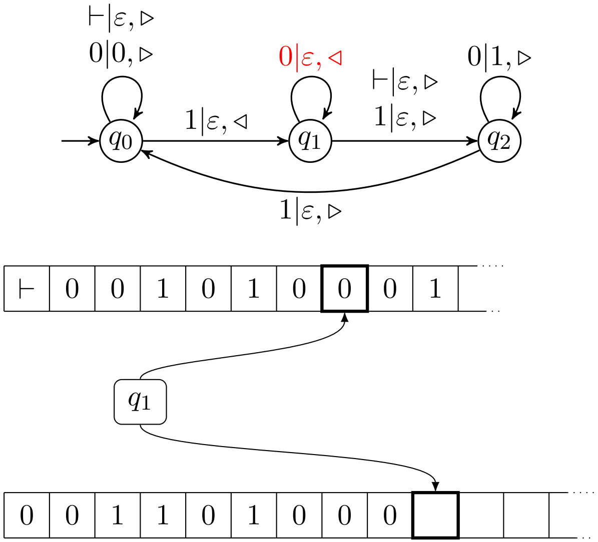

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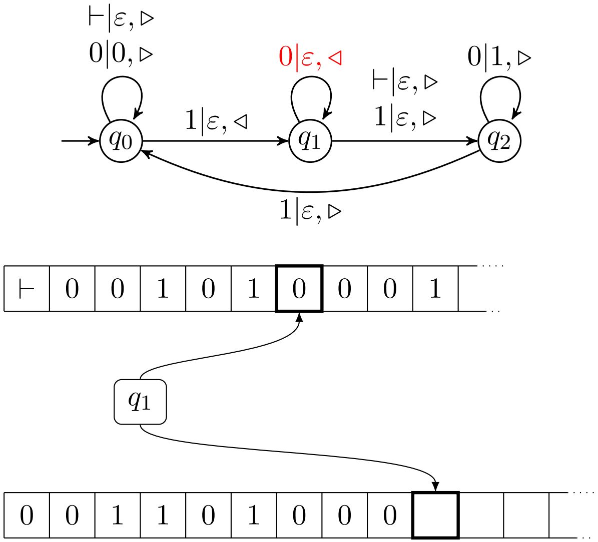

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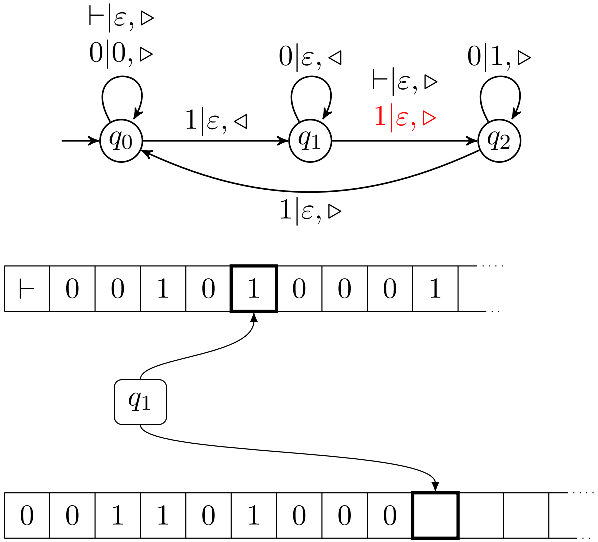

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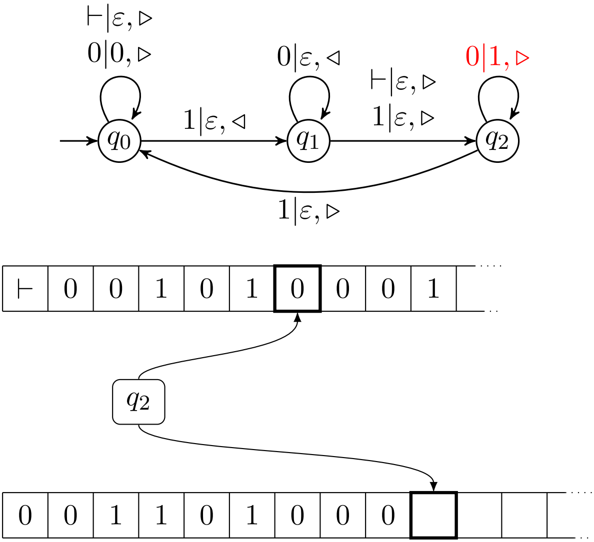

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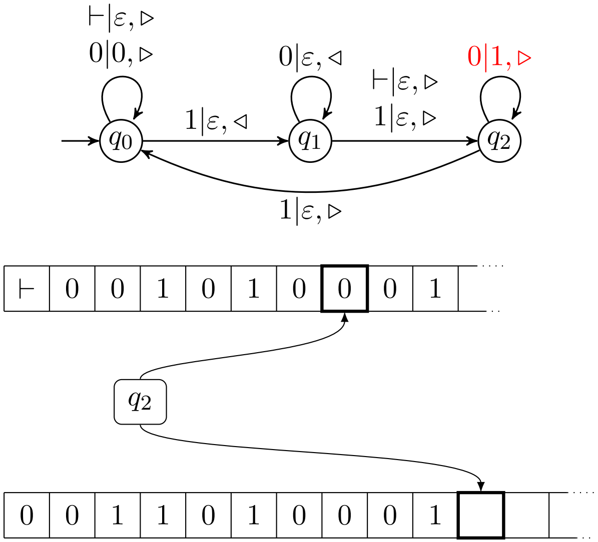
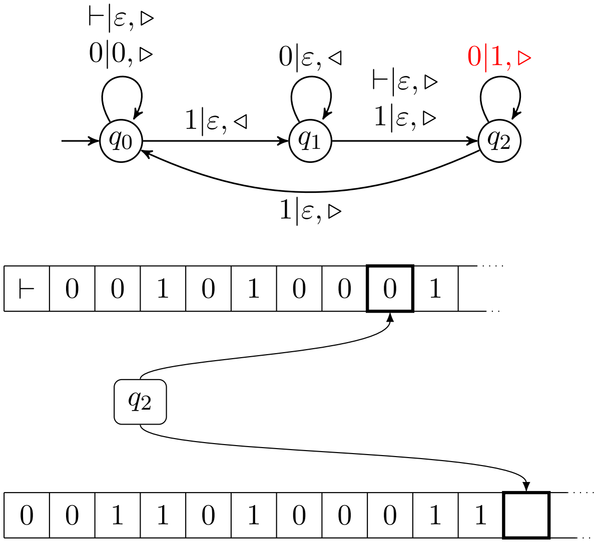
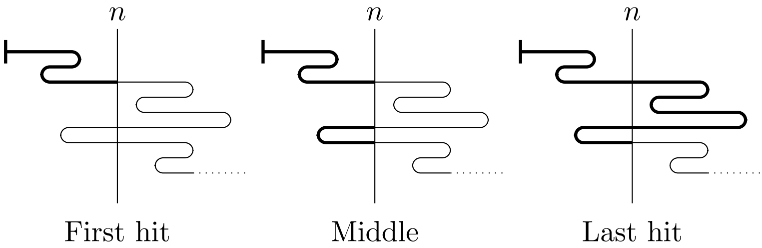


Example: $0^{n_0}10^{n_1}10^{n_2}1\cdots \mapsto 0^{n_0}1^{n_0}0^{n_1}1^{n_1}0^{n_2}1^{n_2}\cdots$



Ratios: first hit, last hit and in the middle

$$\liminf_{n \to \infty} \frac{|\ ?\ |}{n} < 1.$$

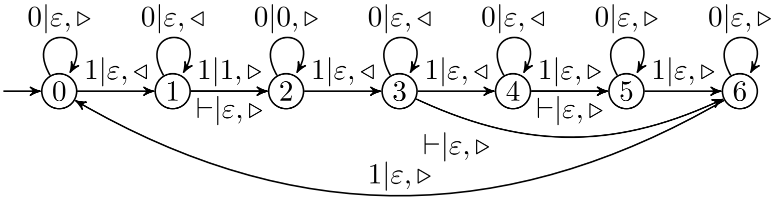


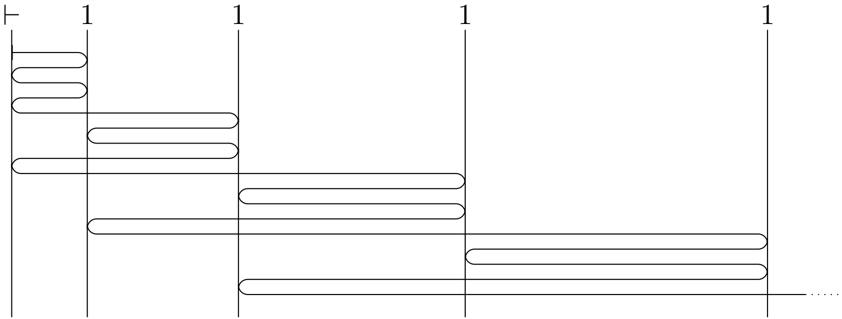
First hit all output made up to the first hit of position n

Middle all output made at positions less than n

Last hit all output made up to the last hit of position n

Different ratios





Two-way transducers cannot compress normal words

Theorem

The first-hit, middle and last-hit ratios of the accepting run of a deterministic bounded-to-one two-way transducer over a normal infinite word coincide.

Theorem

For any run ρ of a non-deterministic two-way bounded-to-one transducer, there is another run ρ' with smaller ratios, such that first-hit, middle and last-hit ratios coincide.

Theorem

Deterministic and non-deterministic two-way bounded-to-one transducers cannot compress normal infinite words.

Selection rules

- If $x = a_1a_2a_3\cdots$ is a normal infinite word, then so is $x' = a_2a_3a_4\cdots$ made of symbols at all positions but the first one.
- If $x = a_1a_2a_3\cdots$ is normal infinite word, then so is $x' = a_2a_4a_6\cdots$ made of symbols at even positions.
- What about selecting symbols at positions 2^n ?
- What about selecting symbols at prime positions ?
- What about selecting symbols following a 1 ?
- What about selecting symbols followed by a 1 ?

Prefix selection

Let $L \subseteq A^*$ be a set of finite words and $x = a_1 a_2 a_3 \cdots \in A^\omega$.
The prefix selection of x by L is the word $x \upharpoonright L = a_{i_1} a_{i_2} a_{i_3} \cdots$
where $\{i_1 < i_2 < i_3 < \cdots\} = \{i : a_1 a_2 \cdots a_{i-1} \in L\}$.

Example (Symbols following a 1)

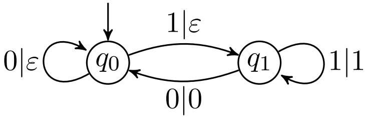
If $L = (0+1)^*1$, then $i_1 - 1, i_2 - 1, i_3 - 1$ are the positions of 1 in x and $x \upharpoonright L$ is made of the symbols following a 1.

Theorem (Agafonov 1968)

Prefix selection by a rational set of finite words preserves normality.

The selection can be realized by a transducer.

Example (Selection of symbols following a 1)



Suffix selection

Let $X \subseteq A^\omega$ be a set of infinite words and $x = a_1 a_2 a_3 \cdots \in A^\omega$. The suffix selection of x by X is the word $x \upharpoonleft X = a_{i_1} a_{i_2} a_{i_3} \cdots$ where $\{i_1 < i_2 < i_3 < \cdots\} = \{i : a_{i+1} a_{i+2} a_{i+3} \cdots \in X\}$.

Example (Symbols followed by a 1)

If $L = 1(0+1)^\omega$, then $i_1 + 1, i_2 + 1, i_3 + 1$ are the positions of 1 in x and $x \upharpoonleft X$ is made of the symbols followed by a 1.

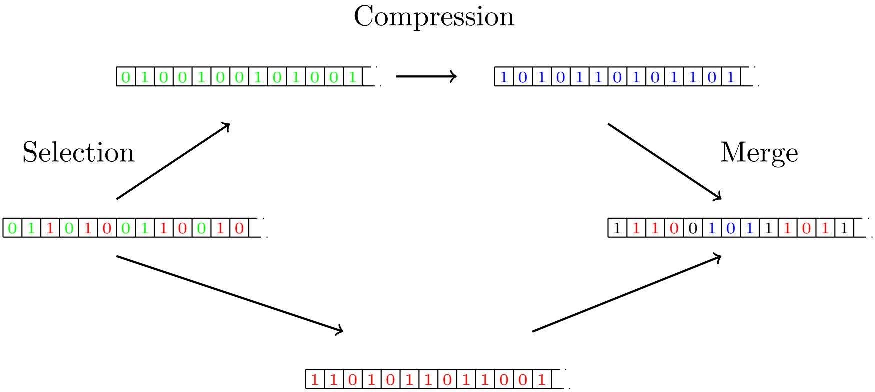
Theorem

Suffix selection by a rational set of infinite words preserves normality.

Ingredients

- transform the selecting transducer into a transducer that splits the input into two infinite words: the selected symbols on one tape and the non-selected symbols on another tape;
- if the word of selected symbols is not normal, use a transducer to compress it;
- use a transducer to merge by blocks the two words into a single one. This expands the output but as little as needed (by increasing the block length)
- combining these transducers gives a bounded-to-one transducer that compresses the input.

Picture


Compression
0 1 0 0 1 0 0 1 0 1 0 0 1
1 0 1 0 1 1 0 1 0 1 1 0 1
Selection
Merge
0 1 1 0 1 0 0 1 1 0 0 1 0
1 1 1 0 0 1 0 1 1 1 0 1 1
1 1 0 1 0 1 1 0 1 1 0 0 1

Combined prefix-suffix selection

Proposition

Let $x = a_1 a_2 a_3 \cdots \in A^\omega$ *be an normal infinite word. The word* $x' = a_{i_1} a_{i_2} a_{i_3} \cdots$ *where* $\{i_1 < i_2 < i_3 < \cdots\} = \{i : a_{i-1} = a_{i+1} = 1\}$ *is not normal.*

Vielen Dank