Around Banach-Mazur games

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The goal of this talk is to present:

my **personal encounter** with **Banach-Mazur games**.

They talk will not reflect an historical perspective[1]!

[1] except from my personal point of view.

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I would like to address the following questions:

- Where, when and how did I discover Banach-Mazur games ?
- Why should you fall in love with them ? (as I already did)

[1] except from my personal point of view.

Outline

Computer programming and software bugs

Computer programming is a difficult task which is error-prone

Definition

A software bug is an error, a failure in a computer program or system that induces an incorrect result.

Bug example: In August 2005, a Malaysian Airlines Boeing 777 that was on autopilot suddenly ascended 3,000 feet.

No need to argue that software without bugs are highly desirable...

A possible solution to automatically check correctness:
model-checking

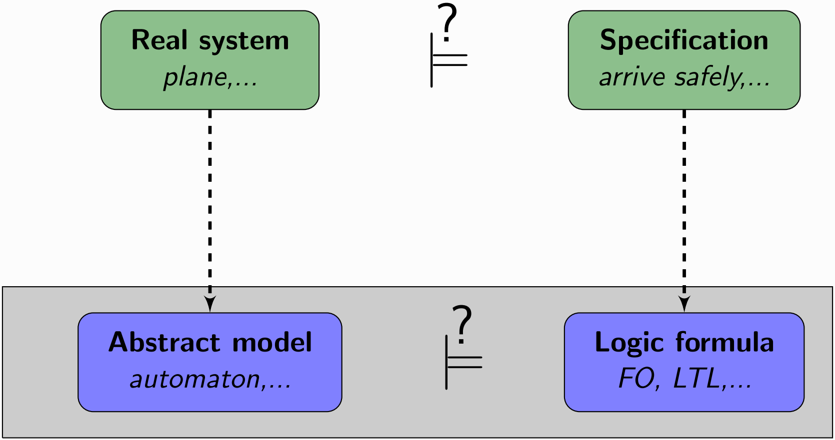
The model-checking picture

Real system
plane,...

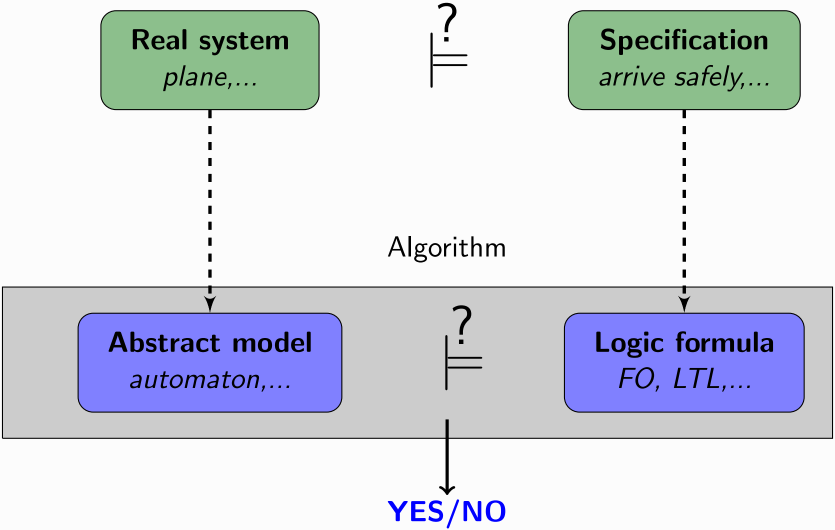
$\stackrel{?}{\models}$

Specification
arrive safely,...

The model-checking picture



The model-checking picture



Model-checking - A 'concrete' example

A faulty coffee/tea machine

Every coffee request provides a coffee

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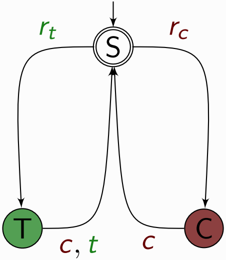

$\mathcal{A}_{Syst}$

$$\varphi_c \equiv \mathbf{G}(r_c \Rightarrow \mathbf{X} c)$$

Model-checking - An important result

How to check 'efficiently' whether $\mathcal{A}_{Syst} \models \varphi_c$?

Theorem [VW86]

Every (LTL) formula can be translated into an equivalent automaton.

[VW86] M. Y. Vardi, P. Wolper: An Automata-Theoretic Approach to Automatic Program Verification. LICS 1986: 332-344.

$$\mathcal{A}_{Syst} \models \varphi_c \quad \text{iff} \quad \mathcal{L}(\mathcal{A}_{Syst}) \subseteq \mathcal{L}(\mathcal{A}_{\varphi_c})$$

$$\text{iff} \quad \mathcal{L}(\mathcal{A}_{Syst}) \cap \mathcal{L}^c(\mathcal{A}_{\varphi_c}) = \emptyset$$

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  - The paper [VV06] about **fair model-checking** was presented.

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With Patricia, we decided to work on *fair model-checking for TA*

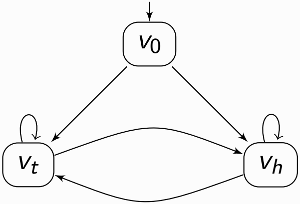
The coin example

Some limits of the classical model-checking approach

Classical Model-Checking

Given a model M and a property φ, decide whether:

$$M \models \varphi, \quad \text{i.e. } \{\rho \text{ execution of } M \mid \rho \not\models \varphi\} \text{ is } \textbf{empty}.$$



$M_{coin} \not\models \mathbf{F}\ \text{head} \quad ; \quad M_{coin} \not\models \mathbf{GF}\ \text{tails}$

Fair model-checking

Fair Model-Checking

Given a model M and a property φ, decide whether:

$$M \mathrel{|}\!\!\approx \varphi, \quad \text{i.e. } \{\rho \text{ execution of } M \mid \rho \not\models \varphi\} \text{ is } \textbf{“very small”}$$

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How to formalise the **fair model-checking** ?

Maybe the most natural answer: via probability

$$M \mathrel{|\!\!\approx}_{\mathbb{P}} \varphi \quad \textit{iff} \quad \mathbb{P}(\{\rho \text{ of } M \mid \rho \not\models \varphi\}) = 0$$
$$\textit{iff} \quad \mathbb{P}(\{\rho \text{ of } M \mid \rho \models \varphi\}) = 1$$

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... in $(\mathbb{R}, |\cdot|)$, we have that $\mathbb{Q}$ is dense and $\mathbb{R} \setminus \mathbb{Q}$ is dense...

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Could dense sets be the **"very big"** sets ? **No**

"Very small" is **meagre**, i.e. countable union of nowhere dense sets.

"Very big" is **large**, i.e. complements of meagre sets.

Few words on meagre sets and large sets

Definitions

Let (X, τ) be a topological space. A set $W \subseteq X$ is:

- **nowhere dense** if the closure of W has empty interior.
  Examples in $(\mathbb{R}, |\cdot|)$*:* $\{a\}$ *with* $a \in \mathbb{R}$, $\mathbb{Z}$, *the Cantor set,...*

Remark

Nowhere dense sets are not stable under countable union: $\mathbb{Q} = \cup_{q \in \mathbb{Q}}\{q\}$

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Meagre sets are also known as sets of first category.

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- **meagre** if it is a countable union of nowhere dense sets.
- **large** if W^c is meagre.

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Meagre sets are also known as sets of first category.

Remark

Large sets are also known as residual sets.

My first encounter with Banach-Mazur game...

Fair Model-Checking problem - topological version

Given a model M and a property φ, decide (algorithmically) whether:

$$\{\rho \text{ exec. of } M \mid \rho \models \varphi\} \text{ is } \textbf{large}.$$

In other words, we need to check whether

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$\{\rho \text{ exec. of } M \mid \rho \not\models \varphi\}$ is **a countable union of nowhere dense sets**.

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Theorem [Oxtoby57]

Let (X, d) be a complete metric space. Let W be a subset of X.

W is large if and only if
Player 0 has a winning strategy in the associated **Banach-Mazur game**.

[Oxtoby57] J.C. Oxtoby, The BanachMazur game and Banach category theorem, Contribution to the Theory of Games, Volume III, Annals of Mathematical Studies 39 (1957), Princeton, 159–163

Outline

Banach-Mazur games

Definition

A *Banach-Mazur game* $\mathcal{G}$ on a finite graph is a triplet (G, v_0, W) where

- $G = (V, E)$ is a finite directed graph with no deadlock,
- $v_0 \in V$ is the initial state,
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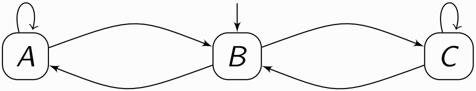
A play $\rho = \rho_1 \rho_2 \rho_3 \cdots$ is won by Pl. 0 wins iff $\rho \in W$.

Banach-Mazur game: an example



$W = \{\ \rho \mid \rho \models \mathbf{GF}\ A \wedge \mathbf{GF}\ C\}$

Example of winning strategy for Pl. 0: $f(\rho) = \begin{cases} BC & \text{if } \rho \text{ ends with } A \\ CBA & \text{if } \rho \text{ ends with } B \\ BA & \text{if } \rho \text{ ends with } C \end{cases}$

A play consistent with f: $\underbrace{BAAA}_{\rho_1}$

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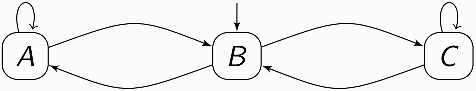

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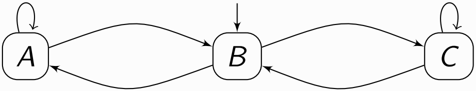

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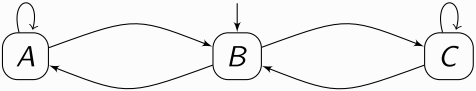

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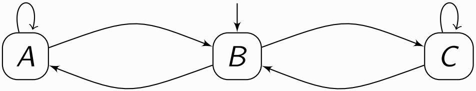

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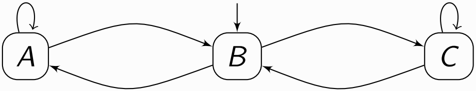

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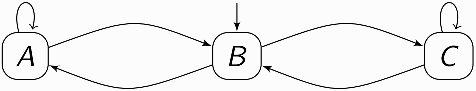

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Banach-Mazur games and large sets

Let (V, E) be a graph, where V^ω equipped with the Cantor topology.

Theorem [Oxtoby57]

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph.

Pl. 0 has a winning strategy for $\mathcal{G}$ if and only if W is large.

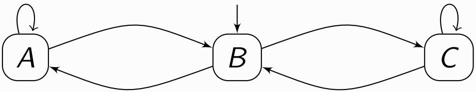
[Oxtoby57] J.C. Oxtoby, The BanachMazur game and Banach category theorem, Contribution to the Theory of Games, Volume III, Annals of Mathematical Studies 39 (1957), Princeton, 159–163

Cantor topology

Given V a finite set, let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be two elements of V^ω.

$$d((a_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}) = 2^{-k} \quad \text{where} \quad k = \min\{i \in \mathbb{N} \mid a_i \neq b_i\}.$$

Banach-Mazur game: an example



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Thus W is a large set.

About determinacy (1)

Theorem [Oxtoby57]

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph.

- Pl. 0 has a winning strategy for $\mathcal{G}$ if and only if W is large.
- Pl. 1 has a winning strategy for $\mathcal{G}$ if and only if W is meagre in some basic open set.

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Corollary

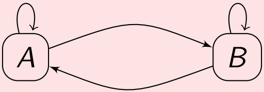
Banach-Mazur games with Borel winning conditions are determined.

1. **Proof 1:** Borel sets have the Baire property (i.e. their symmetric difference with some open set is meagre).
2. **Proof 2:** See Banach-Mazur games as *"classical games played on graphs"* and use the determinacy result from [Ma75].

[Ma75] Donald A. Martin, Borel determinacy. Annals of Mathematics, 1975, Second series 102 (2): 363371

About determinacy (2)

A Banach-Mazur game which is not determined



$$W = \{\ \rho \mid \{i \in \mathbb{N} \mid \rho[i] = A\} \in \mathcal{U}\},$$

where $\mathcal{U}$ is a free ultrafilter.

Ultrafilter on $\mathbb{N}$

A set $\mathcal{U} \subseteq 2^{\mathbb{N}}$ is an ultrafilter on $\mathbb{N}$ if and only if:

- $\emptyset \notin \mathcal{U}$, $\mathcal{U}$ is closed under intersection and supersets,
- for all $S \subseteq \mathbb{N}$, $S \in \mathcal{U}$ or $S^c \in \mathcal{U}$.

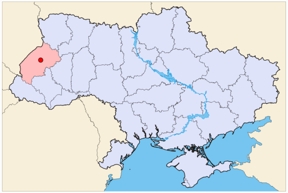
$\mathcal{U}$ is **free** if it contains all co-finite sets (and thus no finite sets).

The **axiom of choice** guarantees existence of free ultrafilter.

Outline

The historical origin of the Banach-Mazur game

In the 1930's and the 1940's, in Lwów (now Lviv in Ukraine)...



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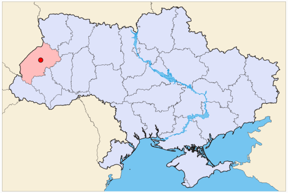

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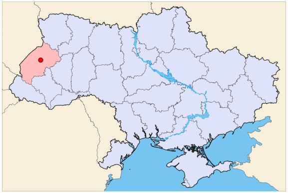

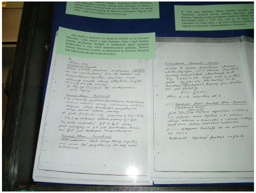

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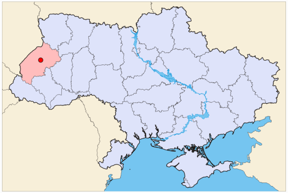

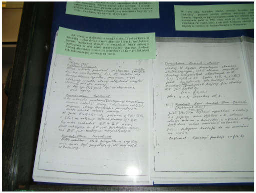

The Scottish book was a note book used by the mathematicians of the *Lwów School of Mathematics* to exchange problems meant to be solved.

The Lwów School of Mathematics

Zjazd Kół Matematyczno-Fizycznych (Lwów 1930). 1 — L. Chwistek, 2 — S. Banach, 3 — S. Loria, 4 — K. Kuratowski, 5 — S. Kaczmarz, 6 — J. P. Schauder, 7 — M. Stark, 8 — K. Borsuk, 9 — E. Marczewski, 10 — S. Ulam, 11 — A. Zawadzki, 12 — E. Otto, 13 — W. Zonn, 14 — M. Puchalik, 15 — K. Szpunar

Problem 43 of the Scottish book

Problem 43 posed by S. Mazur

Definition of a game: Given a set $W \subseteq \mathbb{R}$, Pl. 0 and Pl. 1 alternates in choosing real intervals (starting with Pl. 1) such that:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

A play is won by Pl. 0 if and only if $\cap_{k \geqslant 1} I_k \cap W \neq \emptyset$.

Conjecture: (Price a bottle of wine) W is large if and only if Player 0 has a winning strategy in the above game.

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August 4, 1935

S. Banach: *"Mazur's conjecture" is true*

apparently, without a proof...

Let's play Banach-Mazur games!

- $W = \mathbb{R}$. Clearly $\mathbb{R}$ is large. Thus Pl. 0 has a winning strategy...

  Is any strategy of Pl. 0 winning?

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- $W = \mathbb{R} \setminus \mathbb{Q}$. Let $(q_n)_{n \geqslant 1}$ be an enumeration of $\mathbb{Q}$.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots \supseteq I_k = (a, b)$$

  Given $n_{a,b} := \min\{n \geqslant 1 : q_n \in (a, b)\}$, Pl. 0 can play:

$$(a', b') \quad \text{such that} \quad a < a' < b' < b \quad \text{and} \quad q_{n_{a,b}} \notin (a', b').$$

Outline

A very nice result

A natural question

Given a model M and property φ, do we have that

$$M \mathrel{|\!\approx_{\mathbb{P}}} \varphi \quad \Leftrightarrow \quad M \mathrel{|\!\approx_{T}} \varphi \quad ?$$

In other words, given a set W, do we have that

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Theorem [VV06]

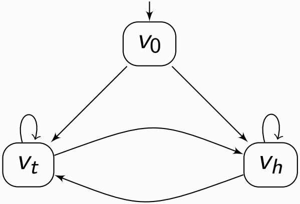
Given a finite system M and an ω-regular property φ, we have that

$$M \approx\!\!\!\!\models_{\mathbb{P}} \varphi \quad \Leftrightarrow \quad M \approx\!\!\!\!\models_{T} \varphi,$$

for bounded Borel measures.

[VV06] D. Varacca, H. Völzer: Temporal Logics and Model Checking for Fairly Correct Systems. LICS 2006: 389-398

How to associate probability distribution with a graph ?



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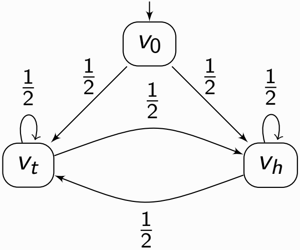


We consider it as a finite Markov chain with uniform distributions.

Remark

The result presented are independent of the probability distributions, as soon as every edge is assigned a positive probability.

Outline

Disturbing phenomena

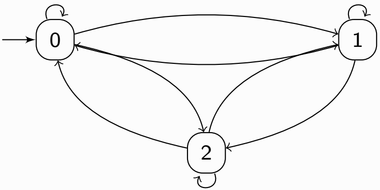
From [VV06], we have that given an ω-regular set W:

$$W \text{ is large } \text{ if and only if } \mathbb{P}(W) = 1,$$

for bounded Borel measures.

Nevertheless, there exists large sets of probability 0...

A large set of probability 0



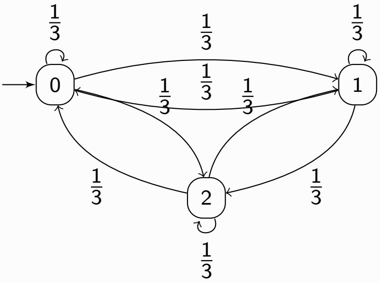
$$W = \{(w_i w_i^R)_i : w_i \in \{0,1,2\}^*\}$$

Pl. 0 has a winning strategy:

$$f(\rho_1 \rho_2 \cdots \rho_{2n+1}) = \rho_{2n+1}^R$$

$\rightsquigarrow$ W is large.

A large set of probability 0



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$\rightsquigarrow$ W is large.

$$\mathbb{P}(W) \leqslant \sum_{n=1}^{\infty} \mathbb{P}(\{w \in W \mid \text{ the first palindrome has length } 2n\})$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(\{w \in \{0,1,2\}^\omega \mid \text{ the first palindrome has length } 2n\}) \cdot \mathbb{P}(W)$$

$$\leqslant \sum_{n=1}^{\infty} \frac{\mathbb{P}(W)}{3^n} = \frac{\mathbb{P}(W)}{2} \qquad \rightsquigarrow \qquad \mathbb{P}(W) = 0 \text{ !!!}$$

There are **large** sets W such that $\mathbb{P}(W) = 0$...

There are **meagre** sets W such that $\mathbb{P}(W) = 1$...

These examples can be very simple (open or closed) sets...

Similarities between meagre sets and negligible sets

$$\mathcal{M} = \{W \subseteq [0,1] \mid W \text{ is meagre}\} \quad ; \quad \mathcal{N} = \{W \subseteq [0,1] \mid \mathbb{P}(W) = 0\}$$

Given $\mathcal{F} = \mathcal{M}$ or $\mathcal{N}$,

1. for any $A \in \mathcal{F}$, if $B \subset A$ then $B \in \mathcal{F}$;
2. for any $(A_n)_{n\geqslant 1} \subset \mathcal{F}$, $\bigcup_{n\geqslant 1} A_n \in \mathcal{F}$;
3. each countable set in $[0,1]$ belongs to $\mathcal{F}$;
4. if $A \in \mathcal{F}$, then $A^c \notin \mathcal{F}$;
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Theorem (Sierpinski, 1920)

Under the continuum hypothesis, there is a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $W \subset \mathbb{R}$ is meagre if and only if $f(W)$ has Lebesgue measure zero.

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But the concepts remains different !!!

[Oxtoby 1971] John C. Oxtoby, Measure and category. A survey of the analogies between topological and measure spaces. Graduate Texts in Mathematics, Vol. 2. Springer-Verlag, New York-Berlin, 1971

Why does it work for ω-regular sets?

Theorem [VV06]

Given a finite system M and an ω-regular property φ, we have that

$$M \approx\!\!\!\models_{\mathbb{P}} \varphi \quad \Leftrightarrow \quad M \approx\!\!\!\models_{T} \varphi,$$

for bounded Borel measures.

The key ingredient to prove the above result is the following result:

Theorem [BGK03]

Given $\mathcal{G} = (G, v_0, W)$ where W is an ω-regular property, we have that

Pl. 0 has a winning strategy for $\mathcal{G}$
iff
Pl. 0 has a **positional** winning strategies for $\mathcal{G}$.

[BGK03] D. Berwanger, E. Grädel, S. Kreutzer: Once upon a Time in a West - Determinacy, Definability, and Complexity of Path Games. LPAR 2003: 229-243

If W is large and ω-regular, then $\mathbb{P}(W) = 1$

Sketch of proof

By [BGK03], Pl. 0 has a **positional winning** strategy f for W on M.
In particular, there is $k \in \mathbb{N}$ such that for all finite prefixes π: $|f(\pi)| \leqslant k$.

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We now see M as a finite Markov chain with uniform distribution.
There is $p > 0$ such that for all finite paths π: $\mathbb{P}(\pi \cdot f(\pi) | \pi) \geqslant p$.

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By means of **Borel-Cantelli Lemma**, we thus have that

$$\mathbb{P}(\{\rho \mid \underbrace{\rho \text{ is a play consistent with } f \text{ on infinitely many prefixes}}_{\rho \text{ is consistent with } f}\}) = 1$$

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As f is winning: $\{\rho \mid \rho \text{ is a play consistent with } f\} \subseteq W$, thus $\mathbb{P}(W) = 1$.

If W is ω-regular and not large, then $\mathbb{P}(W) < 1$

Sketch of proof

Pl. 0 does not have a winning strategy in the BM game $G = (V, v_0, W)$.
By **determinacy**, Pl. 1 has a winning strategy f_1 in G (as W is ω-regular).

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By **determinacy**, Pl. 1 has a winning strategy f_1 in G (as W is ω-regular).

Let π_1 be the first move of Pl. 1 given by f_1. We have that $\mathbb{P}(\pi_1) > 0$.
Notice that f_1 is a winning strategy for Pl. 0 in $G' = (V, \pi_1, W^c)$.

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By the previous implication, we have that

$$\mathbb{P}(W^c \mid \pi_1) = 1.$$

And thus

$$\mathbb{P}(W) < 1.$$

Outline of the talk

Simple strategies for Banach-Mazur games

Given $\mathcal{G} = (G, v_0, W)$, let f be a strategy for Pl. 0.

$$f(\underbrace{\rho_1 \rho_2 \cdots \rho_{2n+1}}_{\text{What is observed}}) = \underbrace{\rho_{2n+2}}_{\text{What is played}}$$

We say that f is

[GL12] E. Grädel, S. Leßenich, *Banach-Mazur Games with Simple Winning Strategies*, CSL 2012

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- **b-bounded** if $|\rho_{2n+2}| \leqslant b$.

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- **move-counting** if it only depends on Last(ρ_{2n+1}) and the number of moves already played.
- **length-counting** if it only depends on the Last(ρ_{2n+1}) and the length of the prefix already played.

[GL12] E. Grädel, S. Leßenich, *Banach-Mazur Games with Simple Winning Strategies*, CSL 2012

About Simple strategies for Pl. 0 (1)

Theorem [BGK03]

Given $\mathcal{G} = (G, v_0, W)$ on a finite graph, we have that

Pl. 0 has a **positional** winning strategy for $\mathcal{G}$

iff

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[BGK03] D. Berwanger, E. Grädel, S. Kreutzer: Once upon a Time in a West - Determinacy, Definability, and Complexity of Path Games. LPAR 2003: 229-243

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[BGK03] E. Grdel, Banach-Mazur Games on Graphs. FSTTCS 2008: 364-382

About Simple strategies for Pl. 0 (2)

Simple observation

Given $\mathcal{G} = (G, v_0, W)$ on a finite graph, we have that

If Pl. 0 has a **positional** winning strategy for $\mathcal{G}$,
then
Pl. 0 has a **bounded** winning strategies for $\mathcal{G}$.

Theorem [BM13,BHM15]

Given $\mathcal{G} = (G, v_0, W)$ on a finite graph, we have that

Pl. 0 has a **length-counting** winning strategy for $\mathcal{G}$
iff
Pl. 0 has a winning strategies for $\mathcal{G}$.

[BM13] T. Brihaye, Q. Menet: Fairly Correct Systems: Beyond omega-regularity. GandALF 2013: 21-34

[BHM15] T. Brihaye, A. Haddad, Q. Menet: Simple strategies for Banach-Mazur games and sets of probability 1, accepted in Information and Computation.

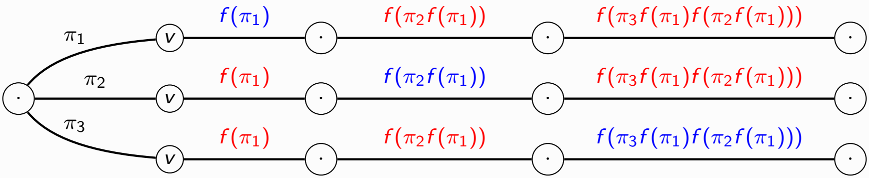
Building a length-counting winning strategy

Sketch of proof

Let f be a winning strat., we have to build $h : V \times \mathbb{N} \to V^*$.

Assume that $\{\pi_1, \pi_2, \pi_3\}$ is the set **finite set of** paths of length n ending in v, then we define:

$$h(v, n) = f(\pi_1) f(\pi_2 f(\pi_1)) f(\pi_3 f(\pi_1) f(\pi_2 f(\pi_1)))$$



If ρ is consistent with h, then ρ is consistent with f (which is winning).

$\rightsquigarrow$ h is a length-counting winning strategy for Pl. 0.

Simple strategies for Pl. 0 on finite graphs

Combining results from [BGK03], [VV06], [G08], [GL12], [BHM15].

Relations with the sets of probability one

Proposition

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

If Pl. 0 has $\begin{cases} \text{a move-counting} \\ \text{a bounded} \end{cases}$ winning strategy for $\mathcal{G}$, then $\mathbb{P}(W) = 1$.

There exist large **open** set of probability 1 without a positional/ bounded/ move-counting winning strategy.

$$W = \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid \exists n > 1 \; w_{n!} = 1\}$$

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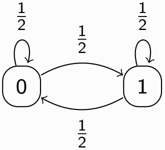
We look for a new concept of "simple strategy"

Back to the example



$W = \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid \exists n > 1 \ \ w_{n!} = 1\}$

Clearly Pl. 0 has a winning strategy, thus W is **large**.

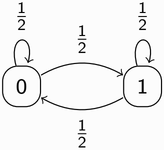
Moreover, we have that $\mathbb{P}(W) = 1$. Indeed, for $n > 1$:

$$A_n := \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid w_{n!} = 1 \text{ and } w_{m!} = 0 \text{ for any } 1 < m < n\},$$

we thus have:

$$W = \dot{\bigcup}_{n>1} A_n \quad \text{and} \quad \mathbb{P}(A_n) = \frac{1}{2^{n-1}} \qquad \rightsquigarrow \qquad \mathbb{P}(W) = 1.$$

Back to the example



$W = \{(w_k)_{k \geqslant 1} \in \{0,1\}^\omega \mid \exists n > 1 \;\; w_{n!} = 1\}$

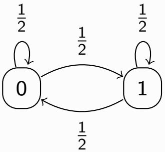
Let f be a b-**bounded** strategy for Pl. 0.

A winning strategy for Pl. 1 (against f) consists in

- starting by playing $(b+1)!$ zeros,
- at each step, completing the sequence by 0's to reach the next $k!$

$\rightsquigarrow$ there is no winning bounded (resp. positional) strategy for Pl. 0.

Back to the example



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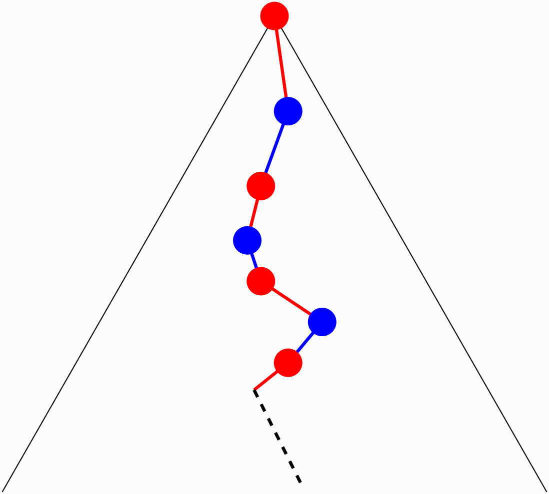
One can also prove the non existence of winning move-counting strategy

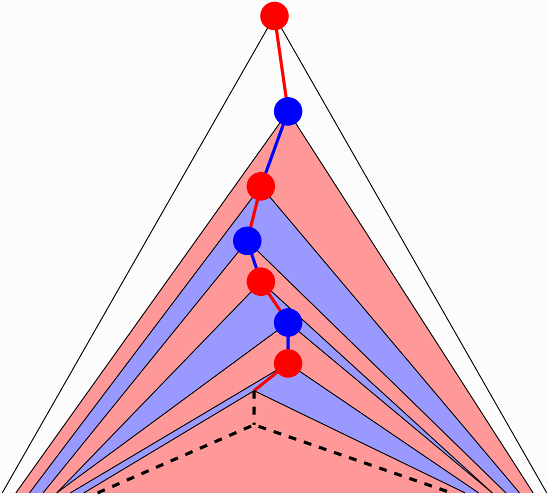
Banach-Mazur game

A play consists in concatenating **finite paths**,



Banach-Mazur game

A play consists in concatenating **finite paths**,
or equivalently in building a decreasing sequence of **open sets**.



Another simple strategy

Given $\mathcal{G} = (G, v_0, W)$, a strategy for Pl. 0 can be seen as $f : \mathcal{O}^* \to \mathcal{O}$.

$$f(\underbrace{O_1 O_2 \cdots O_{2n+1}}_{\text{What is observed}}) \quad = \quad \underbrace{O_{2n+2}}_{\text{What is played}} \quad ,$$

where $O_1 \supseteq O_2 \supseteq \cdots \supseteq O_{2n+1} \supseteq O_{2n+2}$ are open sets.

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Assuming that G is equipped with a probability distribution on edges.

The notion of α-strategy

Given $0 < \alpha < 1$, we say that f is an *α-strategy* if and only if

$$\mathbb{P}\left(O_{2n+2} | O_{2n+1}\right) \geqslant \alpha.$$

Results on α-strategies

Theorem [BM13,BHM15]

Let $\mathcal{G} = (G, v_0, W)$ be a Banach-Mazur game on a finite graph and $\mathbb{P}$ a reasonable probability measure.

If Pl. 0 has a winning α-strategy for some $\alpha > 0$, then $\mathbb{P}(W) = 1$.

Theorem [BM13,BHM15]

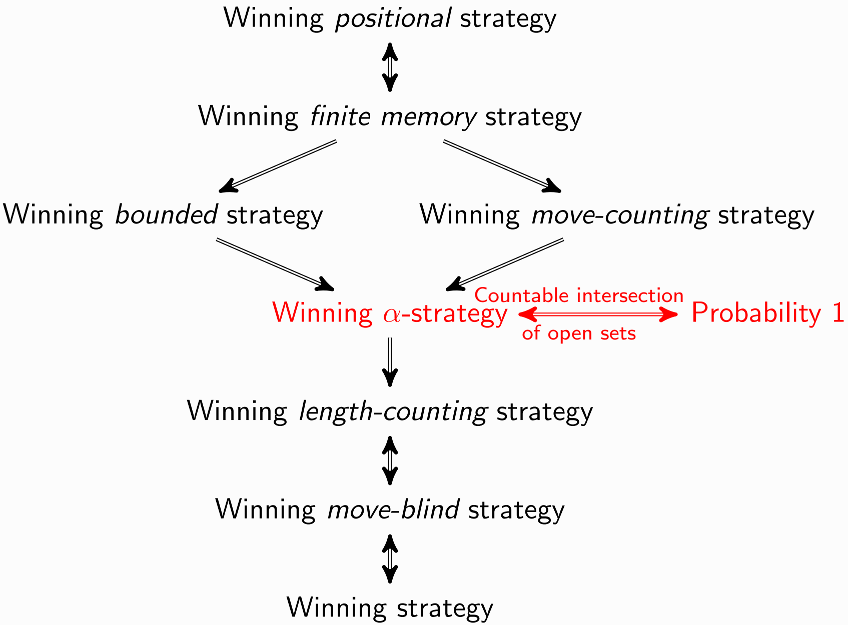
When W is a **countable intersection of open sets**, the following assertions are equivalent:

1. $P(W) = 1$,
2. Pl. 0 has a winning α-strategy for some $\alpha > 0$,
3. Pl. 0 has a winning α-strategy for all $0 < \alpha < 1$.

[BM13] T. Brihaye, Q. Menet: Fairly Correct Systems: Beyond omega-regularity. GandALF 2013: 21-34

[BHM15] T. Brihaye, A. Haddad, Q. Menet: Simple strategies for Banach-Mazur games and sets of probability 1, accepted in Information and Computation.

Summary


Winning *positional* strategy
Winning *finite memory* strategy
Winning *bounded* strategy
Winning *move-counting* strategy
Winning α-strategy
Countable intersection
of open sets
Probability 1
Winning *length-counting* strategy
Winning *move-blind* strategy
Winning strategy

Abour fair model-checking of timed automata (1)

Theorem [BBB+14]

Given a timed automaton $\mathcal{A}$ and an ω-regular property φ, we have that

$$\mathcal{A} \mid\!\approx_{\mathbb{P}} \varphi \quad \Leftrightarrow \quad \mathcal{A} \mid\!\approx_{\mathcal{T}} \varphi,$$

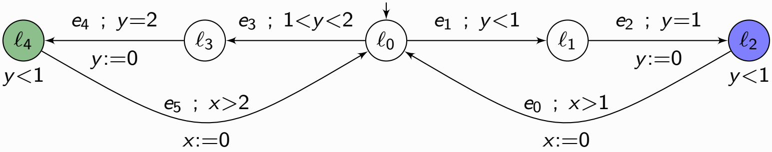
in the following cases:

- if φ is a safety property.
- if $\mathcal{A}$ is a one-clock timed automaton.
- if $\mathcal{A}$ is a reactive timed automaton.

[BBB+14] Nathalie Bertrand, Patricia Bouyer, Thomas Brihaye, Quentin Menet, Christel Baier, Marcus Groesser, Marcin Jurdzinski: Stochastic Timed Automata. Logical Methods in Computer Science 10(4) (2014)

Abour fair model-checking of timed automata (2)

The previous theorem is false in general:



Let φ be the formula **GF** ℓ_2, we have that

$$\mathcal{A} \mathrel{|}\!\!\approx_T \varphi \quad \text{but} \quad \mathcal{A} \not\mathrel{|}\!\!\approx_{\mathbb{P}} \varphi.$$

Let y_n be the value of y at the n^{th} arrival in ℓ_0

$$y_n < 1 \qquad \text{and} \qquad y_n < y_{n+1}$$

Conclusion

Why should you fall in love with Banach-Mazur games?

- They are fun!
- They enjoy nice properties (positional strategies suffice for ω-regular winning conditions).
- They help understanding topological concepts.
- The study of their winning strategy helps in understanding links between topological bigness and probabilistic bigness.
- ...

Thank you!!!